

Minimal Bar Tableaux

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Abstract. Motivated by Stanley's results in [7], we generalize the rank of a partition λ to the rank of a shifted partition $S(\lambda)$. We show that the number of bars required in a minimal bar tableau of $S(\lambda)$ is $\max(o, e + (\ell(\lambda) \bmod 2))$, where o and e are the number of odd and even rows of λ . As a consequence we show that the irreducible projective characters of S_n vanish on certain conjugacy classes. Another corollary is a lower bound on the degree of the terms in the expansion of Schur's Q_λ symmetric functions in terms of the power sum symmetric functions.

Keywords: bar tableau, strip tableau, rank, shifted partition, shifted shape, projective character, negative character, Schur Q-functions

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of the integer n , i.e., $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\sum \lambda_i = n$. The *length* $\ell(\lambda)$ of a partition λ is the number of nonzero parts of λ . The (Durfee or Frobenius) *rank* of λ , denoted $\text{rank}(\lambda)$, is the length of the main diagonal of the diagram of λ , or equivalently, the largest integer i for which $\lambda_i \geq i$. The rank of λ is the least integer r such that λ is a disjoint union of r border strips (also called ribbons or rim hooks). Denote by $\chi^\lambda(\pi)$ the character of the irreducible representation of S_n indexed by λ evaluated at a permutation of cycle type π . An easy application of the Murnaghan-Nakayama rule shows that if $\ell(\pi) < \text{rank}(\lambda)$, we must have $\chi^\lambda(\pi) = 0$. As a corollary, the expansion of the Schur function in terms of the power sum symmetric functions

$$s_\lambda = \sum_{\pi} \chi^\lambda(\pi) \frac{p_\pi}{z_\pi}$$

contains only terms $\chi^\lambda(\pi) \frac{p_\pi}{z_\pi}$ such that $\ell(\pi) \geq \text{rank}(\lambda)$.

We generalize the notion of rank to shifted diagrams $S(\lambda)$ of a partition with distinct parts by counting the minimal number of bars in a bar tableau. Using Morris' projective analogue of the Murnaghan-Nakayama rule, we show that the irreducible projective characters of S_n vanish on conjugacy classes indexed by partitions with few parts. This enables us to give a lower bound on the length of the μ which appear in the expansion of the Schur Q-functions in terms of the p_μ .

2. Bar Tableaux

Let $\mathcal{D}(n)$ be the set of all partitions of n into distinct parts. The *shifted diagram*, $S(\lambda)$, of shape λ is obtained by forming l rows of nodes with λ_i nodes in the i th row such that, for all $i > 1$, the first node in row i is placed underneath the second node in row $(i - 1)$. For instance Figure 1 shows the shifted diagram of the shape 97631.

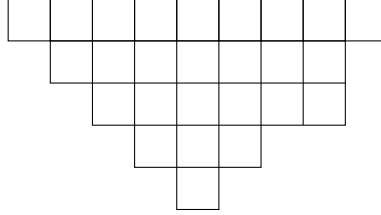


Figure 1: The shifted diagram of the shape 97631.

We follow the treatment of Hoffman and Humphreys [2] to define bar tableaux. These occur in the inductive formula for the projective characters of S_n , first proved by Morris [5]. Let r be an *odd* positive integer, and let $\lambda \in \mathcal{D}(n)$ have length l . Below we define:

- (1) a subset, $I_+ \cup I_0 \cup I_- = I(\lambda, r)$, of integers between 1 and l ; and
- (2) for each $i \in I(\lambda, r)$, a strict partition $\lambda(i, r)$ in $\mathcal{D}(n - r)$ (despite the notation, $\lambda(i, r)$ is a function of λ , as well as of (i, r)).

Let

$$I_+ = \{i: \lambda_{j+1} < \lambda_i - r < \lambda_j \text{ for some } j \leq l, \text{ taking } \lambda_{l+1} = 0\}.$$

In other words I_+ is the set of all rows of λ which we can remove r squares from and still leave a composition with distinct parts. For example, if $r = 5$ and $\lambda = 97631$, then $I_+ = \{1, 2\}$. If $i \in I_+$, then $\lambda_i > r$, and we define $\lambda(i, r)$ to be the partition obtained from λ by removing λ_i and inserting $\lambda_i - r$ between λ_j and λ_{j+1} . Continuing our example above, $\lambda(2, 5) = 96321$. Let

$$I_0 = \{i: \lambda_i = r\},$$

which is empty or a singleton. For $i \in I_0$, remove λ_i from λ to obtain $\lambda(i, r)$. Let

$$I_- = \{i: r - \lambda_i = \lambda_j \text{ for some } j \text{ with } i < j \leq l\}.$$

Equivalently I_- is the set of all rows of λ for which there is some shorter row of λ such that the total number of squares in both rows is r . For example, if $r = 7$ and $\lambda = 97631$, then $I_- = \{3\}$. If $i \in I_-$, then $\lambda_i < r$, and $\lambda(i, r)$ is formed by removing both λ_i and λ_j from λ .

For each $i \in I(\lambda, r)$ the associated r -bar is given as follows. If i is in I_+ or I_0 , the r -bar consists of the rightmost r nodes in the i th row of $S(\lambda)$. We say the r -bar is of *type 1* or *type 2* respectively. For example, the squares in Figure 2 labelled by 6 are a 7-bar of type 1. The squares labelled by 4 are a 3-bar of type 2. If i is in I_- , the r -bar consists

of all the nodes in both the i th and j th rows, a total of r nodes. We say the r -bar is of *type 3*. The squares in Figure 2 labelled by 3 are a 7-bar of type 3.

Define a *bar tableau* of shape λ to be an assignment of positive integers to the squares of $S(\lambda)$ such that

- (1) the set of squares occupied by the biggest integer is an r -bar B , and
- (2) if we remove the r -bar B and reorder the rows, the result is a bar tableau.

1	1	6	6	6	6	6	6	6
	1	2	2	2	5	5	5	
		3	3	3	3	3	3	
			4	4	4			
				3				

Figure 2: A bar tableau of the shape 97631.

Equivalently we can define a *bar tableau* of shape λ to be an assignment of positive integers to the squares of $S(\lambda)$ such that

- (1) the entries are weakly increasing across rows,
- (2) each integer i appears an odd number of times,
- (3) i can appear in at most two rows; if it does, it must begin both rows (equivalent to the bar being of type 3),
- (4) the composition remaining if we remove all squares labelled by integers larger than some i has distinct parts.

For example, Figure 3 shows the chain of partitions remaining if we remove all squares labelled by integers larger than some i from the tableau in Figure 2. This demonstrates the legality of that tableau.

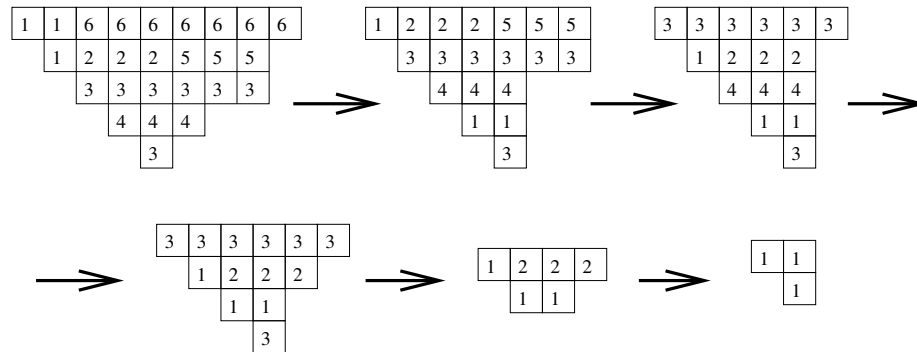


Figure 3: Checking legality of the bar tableau of the shape 97631.

3. Minimal Bar Tableaux

We introduce an operation on minimal bar tableaux which preserves the number of bars, and prove some facts about tableaux resulting from this operation. A bar tableau of λ is *minimal* if the number of bars is minimized, i.e. there does not exist a bar tableau with fewer bars.

Lemma 3.1. *There exists a minimal bar tableau T^* such that there is no bar boundary an even number of squares along any row.*

For example Figure 4 shows a minimal bar tableau T of shape 97631 and a minimal bar tableau T^* of the same shape with no bar boundaries an even number of squares along any row (we will verify later that these tableaux are minimal).

4	4	4	4	4	4	4	4	4
	2	2	2	2	3	3	3	
		1	1	1	1	1	1	
			2	2	2			
				1				

T

4	4	4	4	4	4	4	4	4
	3	3	3	3	3	3	3	
		1	1	1	1	1	1	
			2	2	2			
				1				

T*

Figure 4: Two minimal bar tableaux of shape 97631.

Proof. Let T be a minimal bar tableau of λ . In each row r_k of T , at the last bar boundary an even number of squares along a row, let b be the bar which begins to the right of the boundary. Say that b is labelled by j . Relabel the squares to the left of the boundary with j . This preserves the ordering on labels and the parity of b . The partitions remaining if we remove all squares labelled by integers larger than $i (> j)$ will be the same as before and have distinct parts. The partitions remaining if we remove all squares labelled by $i (< j)$ will not contain row r_k but will otherwise have the same (distinct) parts as before. ■

Lemma 3.2. *Let T^* be a minimal bar tableau of λ such that there is no bar boundary an even number of squares along any row. Then if row r_k is odd, it is labelled entirely by one label j . If row r_k is even, it is labelled entirely by one label j or it has exactly two labels each occurring an odd number of times.*

Proof. If row r_k is odd and has more than one label, the second bar must be of type 1. Therefore the second bar must be odd, forcing the first bar to be even which is a contradiction.

If row r_k is even and has more than two labels, the final two bars are both of type 1 and so must be odd, forcing there to be a bar boundary an even number of squares along the row, a contradiction. ■

4. Number of Bars in a Minimal Tableau

We use the results from the previous section to count how many bars are needed in a minimal bar tableau. Define the *shifted rank* of a shape λ , denoted by $\text{srank}(\lambda)$, to be the number of bars in a minimal bar tableau of λ . Given an integer a , define $a \bmod 2$ to be 1 if a is odd and 0 if a is even.

Theorem 4.1. *Given a shape λ , let o be the number of odd rows of λ and e be the number of even rows. Then $\text{srank}(\lambda) = \max(o, e + (\ell(\lambda) \bmod 2))$.*

For example, if $\lambda = 97631$, we have $o = 4$, $\ell(\lambda) = 5$ and $e = 1$. So $\text{srank}(\lambda) = \max(4, 1 + 1) = 4$ which verifies that the tableaux shown in Figure 4 are indeed minimal. If $\lambda = 432$, we have $o = 1$, $\ell(\lambda) = 3$ and $e = 2$. So $\text{srank}(\lambda) = \max(1, 3) = 3$. Such a tableau is illustrated in Figure 5.

2	3	3	3
	1	1	1
		1	1

Figure 5: A minimal bar tableaux of shape 432.

Proof. Let T be a minimal bar tableau of λ . Preprocess T into T^* so that there are no bar boundaries an even number of squares along any row. This must preserve the number of bars. Bars of type 3 consist of one even initial bar and one odd initial bar, and so by Lemma 3.2 must be an entire even row and an entire odd row, or an entire even row and the initial odd bar of some other even row.

First assume that $o \geq e$. Note that if $o = e$, then $\ell(\lambda) \bmod 2 = 0$. So when $o \geq e$, $\max(o, e + (\ell(\lambda) \bmod 2)) = o$. We claim that the bars of type 3 all consist of entire even row and entire odd row pairs, and that there are exactly e of them.

From the observations above, the number of bars of type 3 cannot be larger than e . Suppose that there is a bar of type 3 consisting of an entire even row and the initial odd bar of some other even row. Since $o \geq e$, there must also be two other odd rows, not parts of bars of type 3, each labelled entirely by some label (by Lemma 3.2). The total number of bars in these 4 rows is 4. So if we relabel (with new large labels) these four rows as two bars of type 3, we save two bars, contradicting the minimality of T^* (note that relabelling entire rows with some integer larger than any current label preserves legality, provided the parities are correct). We illustrate this (impossible) situation in Figure 6, and show the more economical version. Thus there are no such bars of type 3.

Now suppose that the number of bars of type 3 is smaller than e ; thus there is some even row r_1 of the tableau which is not part of a bar of type 3. Also there is an odd row r_2 which is not part of a type 3 bar (since $o \geq e$). But we could relabel both these rows with some new large label saving at least one bar and contradict the minimality of T^* .

So there are exactly e bars of type 3, filling $2e$ rows of λ . The remaining $o - e$ rows are odd and so must each be completely filled by a unique label. So the total number of bars is o as required.

4	4	4	4	4
	1	1	1	1
		1	2	
			3	

8	8	8	8	8
	8	8	8	8
		9	9	
			9	

Figure 6: A minimal and a nonminimal bar tableau of shape 5421.

Now assume that $e \geq o$. First we show that we can relabel so that every odd row is part of a bar of type 3. So suppose r_3 is an odd row which is not part of a bar of type 3.

Claim. There is an even row r_4 , completely filled by a label, which is part of a bar of type 3 with the initial part of some other even row r_5 .

Proof of claim. Assume by way of contradiction that there is not, i.e. that the completely filled even rows are all parts of bars of type 3 with complete odd rows. But there must be at least one even row r_4 (since $e \geq o$ and r_3 is not part of a bar of type 3) which is not part of a bar of type 3 with a complete odd row. So r_4 must not be part of a bar of type 3 at all (by our claim assumption). But then we could relabel r_4 and r_3 entirely with some new large label and save a bar, a contradiction. This proves our claim.

Relabel r_4 and r_3 with some new large label. This leaves an odd number of squares in the initial part of row r_5 , and so preserves legality. These two rows are now a valid bar of type 3, and this process did not cost us any bars. We illustrate one step of this process in Figure 7. Simply iterate this process until there are no odd rows which are not part of bars of type 3. This proves that we can relabel so that every odd row is part of a bar of type 3.

2	2	2	2
	1	1	1
		2	4

9	9	9	9
	9	9	9
		2	4

Figure 7: Relabelling to remove an odd row not part of a bar of type 3.

So every odd row is part of a bar of type 3, filling $2o$ rows of λ . All (except for possibly one) of the even rows remaining must be paired up with another remaining even row, and each pair must contain one bar of type 3 (filling one entire row and the odd initial part of the other row) and one bar of type 1 (filling the odd final part of the other row). If they were not, we would have two even rows costing 4 strips, and could reduce the number of strips by relabelling as above with large new numbers. The extra row exists only when $\ell(\lambda)$ is odd, and costs two bars (i.e. one extra). This situation is illustrated on the right hand side of the above figure. So we have $o + e - o + (\ell(\lambda) \bmod 2)$ strips as required. ■

It remains an open problem to count how many minimal shifted tableau there are for a given shape λ . Also, it would be natural to generalise Theorem 4.1 to the skew

shifted case. It is possible to define skew bar tableaux, but the minimal number of bars therein remains an open question (see [1] for more details).

5. Projective Representations of the Symmetric Group

Here we recall some facts about the projective representations of the symmetric group. We follow the treatment of Stembridge [8].

A projective representation of a group G on a vector space V is a map $P: G \rightarrow GL(V)$ such that

$$P(x)P(y) = c_{x,y}P(xy) \quad (x, y \in G)$$

for suitable (nonzero) scalars $c_{x,y}$. For the symmetric group, the associated Coxeter presentation shows that a representation P amounts to a collection of linear transformations $\sigma_1, \dots, \sigma_{n-1} \in GL(V)$ (representing the adjacent transpositions) such that σ_j^2 , $(\sigma_j\sigma_{j+1})^3$, and $(\sigma_j\sigma_k)^2$ (for $|j-k| \geq 2$) are all scalars. The possible scalars that arise in this fashion are limited. Of course, one possibility is that the scalars are trivial; this occurs in any ordinary linear representation of S_n . According to a result of Schur [6], there is only one other possibility (occurring only when $n \geq 4$); namely

$$\sigma_j^2 = -1; \quad (\sigma_j\sigma_k)^2 = -1 \text{ (for } |j-k| \geq 2\text{)}; \quad (\sigma_j\sigma_{j+1})^3 = -1. \quad (5.1)$$

All other possibilities can be reduced to this case or the trivial case by a change of scale. See [3, 9] for details.

It is convenient to regard $\sigma_1, \dots, \sigma_{n-1}$ as elements of an abstract group, and to take 5.1 as a set of defining relations. More precisely, for $n \geq 1$ let us define \tilde{S}_n to be the group of order $2 \cdot n!$ generated by $\sigma_1, \dots, \sigma_{n-1}$ (and -1), subject to the relations 5.1, along with the obvious relations $(-1)^2 = 1$, $(-1)\sigma_j = \sigma_j(-1)$ which force -1 to be a central involution. By Schur's Lemma, an irreducible linear representation of \tilde{S}_n must represent -1 by either of the scalars $+1$ or -1 . A representation of the former type is a linear representation of S_n , whereas one of the latter type corresponds to a projective representation of S_n as in 5.1. We will refer to any representation of \tilde{S}_n in which the group element -1 is represented by the scalar -1 as a *negative representation* of \tilde{S}_n .

Next we review the characters of the irreducible negative representations of \tilde{S}_n . Define $\mathcal{P}(n)$ to be the set of all partitions of n . We say that a partition λ is *odd* if and only if the number of even parts in λ is odd, and is *even* if and only if it is not odd. Thus, the parity of a permutation agrees with the parity of its cycle type. The parity of λ is also the parity of the integer $|\lambda| + \ell(\lambda)$. Schur showed that the irreducible negative representations are indexed by partitions λ with distinct parts. Recall that if P is an irreducible negative representation indexed by λ that the character $\langle \lambda \rangle$ is a class function $\langle \lambda \rangle: \tilde{S}_n \rightarrow \mathbb{Q}$ defined by $\langle \lambda \rangle(g) = \text{trace}(P(g))$.

If $g = \pm\sigma_{i_1}\sigma_{i_2}\cdots$, let $\pi \in \mathcal{P}(n)$ be the cycle type (in S_n) of $\sigma_{i_1}\sigma_{i_2}\cdots$. In the sequel we will evaluate $\langle \lambda \rangle(\pi)$ instead of $\langle \lambda \rangle(g)$. Define $\mathcal{P}^0(n)$ to be all partitions of n such that all parts are odd.

Theorem 5.1 (Schur 1911 [6]). *Let $\lambda \in \mathcal{D}(n)$ have length ℓ , and let $\pi \in \mathcal{P}(n)$.*

- *Suppose that λ is odd. If π is neither in $\mathcal{P}^0(n)$ nor equal to λ then $\langle \lambda \rangle(\pi) = 0$.*

- Suppose that λ is odd. If π equals λ then

$$\langle \lambda \rangle(\pi) = \pm i^{(n-\ell+1)/2} (\lambda_1 \lambda_2 \cdots \lambda_\ell / 2)^{1/2}.$$

- Suppose that λ is even. If π is not in $\mathcal{P}^0(n)$ then $\langle \lambda \rangle(\pi) = 0$.

For example we consider the situation when $n = 6$ and $\lambda = 321$. Then $\langle \lambda \rangle(\pi) = 0$ when π is (6) , (42) , (411) , (222) , (2211) or (21111) , as these partitions all have one even part. The second fact gives $\langle \lambda \rangle(\pi) = \sqrt{3}$ when $\pi = (321)$. If $\lambda = 51$ then $\langle \lambda \rangle(\pi) = 0$ when π is (6) , (42) , (411) , (321) , (222) , (2211) or (21111) .

A combinatorial rule for calculating the characters not specified by Schur's theorem was given by Morris; it is the projective analogue of the Murnaghan-Nakayama rule.

Theorem 5.2 (Morris 1962 [5]). *Let $\lambda \in \mathcal{D}(n)$ have length ℓ . Suppose that $\pi \in \mathcal{P}^0(n)$ and that π contains r at least once. Define $\pi' \in \mathcal{P}^0(n-r)$ by removing a copy of r from π . Then*

$$\langle \lambda \rangle(\pi) = \sum_{i \in I(\lambda, r)} n_i \langle \lambda(i, r) \rangle(\pi'),$$

where

$$n_i = \begin{cases} (-1)^{j-i} 2^{1-\varepsilon(\lambda)}, & \text{if } i \in I_+, \\ (-1)^{\ell-i}, & \text{if } i \in I_0, \\ (-1)^{j-i+\lambda_i} 2^{1-\varepsilon(\lambda)}, & \text{if } i \in I_-. \end{cases}$$

(The integer j is that occurring in the definitions of I_\pm , and $\varepsilon(\lambda)$ is the parity of λ ; i.e. 0 or 1.)

For example if $n = 6$, $\lambda = (51)$ and $r = 1$, we have $\varepsilon(\lambda) = 0$, $I_+ = \{1\}$, $I_0 = \{2\}$ and $I_- = \emptyset$. So $I(\lambda, r) = \{1, 2\}$, and we have

$$\begin{aligned} \langle 51 \rangle(1^6) &= (-1)^{1-1} 2^{1-0} \langle 41 \rangle(1^5) + (-1)^{2-2} \langle 5 \rangle(1^5) \\ &= 2 \langle 41 \rangle(1^5) + \langle 5 \rangle(1^5). \end{aligned}$$

Expand this sum into a sum over all possible bar tableaux. Define the *weight* of a tableau $\text{wt}(T)$ to be the product of all the powers of -1 and 2 which appear. Then we have

$$\langle \lambda \rangle(\pi) = \sum_T \text{wt}(T),$$

summed over all bar tableaux of shape λ and type π . We know that the shifted rank of λ is the minimum number of bars needed in a bar tableau of shape λ . So we obtain the following result as a corollary to Theorem 4.1:

Corollary 5.3. *Given a shape λ of shifted rank k and a shape π such that $\ell(\pi) < k$, we have $\langle \lambda \rangle(\pi) = 0$.*

6. Schur Q-Functions

We begin with Schur's original inductive definition of the Q_λ functions. Denote the monomial symmetric functions by m_λ and define symmetric functions q_k of degree k by

$$q_k = \sum_{\lambda \in \mathcal{P}(k)} 2^{\ell(\lambda)} m_\lambda.$$

Now we can state the base cases for the inductive definition. Put $Q_{(a)} = q_a$ and

$$Q_{(a,b)} = q_a q_b + 2 \sum_{n>0} (-1)^n q_{a+n} q_{b-n}.$$

Inductively we define

$$Q_{\lambda_1, \dots, \lambda_{2k+1}} = \sum_{i=1}^{2k+1} (-1)^{i+1} q_{\lambda_i} Q_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{2k+1}}$$

and

$$Q_{\lambda_1, \dots, \lambda_{2k}} = \sum_{i=2}^{2k} (-1)^i Q_{\lambda_1, \lambda_i} Q_{\lambda_2, \dots, \hat{\lambda}_i, \dots, \lambda_{2k}}.$$

The Q_λ may also be defined as the specialization at $t = -1$ of one of the two equivalent defining formulae for Hall-Littlewood polynomials; see [4, III (2.1) (2.2)]. Let S_r act on $X = \{x_1, \dots, x_r\}$ by permuting the variables, so that, when $\ell \leq r$, the Young subgroup $S_1^\ell \times S_{r-\ell}$ fixes each of x_1, \dots, x_ℓ . Let λ be a strict partition of length ℓ . If $\ell \leq r$, then

$$Q_\lambda(x_1, \dots, x_r) = 2^\ell \sum_{[w] \in S_r / S_1^\ell \times S_{r-\ell}} w \left\{ x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell} \prod_{i=1}^{\ell} \prod_{j=i+1}^r \frac{x_i + x_j}{x_i - x_j} \right\}.$$

If λ has length greater than r , then $Q_\lambda(x_1, \dots, x_r) = 0$. The Q_λ symmetric functions are obtained by taking the limit as the number of variables becomes infinite (for a mathematically precise definition of this limit see [4]).

Schur [6] defined these Q-functions in order to study the projective representations of symmetric groups. The fundamental connection is given by the following theorem. Let $m_i(\lambda) = \#\{j: \lambda_j = i\}$, the number of parts of λ equal to i . Define $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. Denote the power sum symmetric functions by p_λ .

Theorem 6.1 (Schur, 1911 [6]).

$$Q_\lambda = \sum_{\pi \in \mathcal{P}^0(n)} 2^{[\ell(\lambda) + \ell(\pi) + \varepsilon(\lambda)]/2} \langle \lambda \rangle (\pi) \frac{p_\pi}{z_\pi}.$$

Again consider the example with $n = 6$ and $\lambda = (51)$. We have

$$\begin{aligned} Q_{51} &= 2^{[2+6+0]/2} \langle 51 \rangle (1^6) \frac{P_{1^6}}{z_{1^6}} + 2^{[2+4+0]/2} \langle 51 \rangle (1^3 3) \frac{P_{1^3 3}}{z_{1^3 3}} \\ &\quad + 2^{[2+2+0]/2} \langle 51 \rangle (15) \frac{P_{15}}{z_{15}} + 2^{[2+2+0]/2} \langle 51 \rangle (3^2) \frac{P_{3^2}}{z_{3^2}} \\ &= 2^4 16 \frac{P_{1^6}}{6!} + 2^3 2 \frac{P_{1^3 3}}{3! 3} - 2^2 1 \frac{P_{15}}{5} - 2^2 2 \frac{P_{3^2}}{18} \\ &= \frac{16}{45} P_{1^6} + \frac{8}{9} P_{1^3 3} - \frac{4}{5} P_{15} - \frac{4}{9} P_{3^2}. \end{aligned}$$

Define $\deg(p_i) = 1$, so $\deg(p_{\mathbf{v}}) = \ell(\mathbf{v})$. Then Theorem 4.1 gives us the following corollary.

Corollary 6.2. *The terms of lowest degree in Q_{λ} have degree at least $\text{srank}(\lambda)$.*

In our example $\text{srank}(51) = 2$ and the $p_{\mathbf{v}}$ satisfy $\ell(\mathbf{v}) \geq 2$. Equivalently we can examine a specialization of the principal specialization of Q_{λ} , i.e.

$$ps_t^1(Q_{\lambda}) = Q_{\lambda}(1, \underbrace{1, \dots, 1}_{t \text{ 1's}}) = Q_{\lambda}(1^t).$$

Since $p_{\mathbf{v}}(1^t) = t^{\ell(\mathbf{v})}$, we can rephrase the above result.

Corollary 6.3. *$Q_{\lambda}(1^t)$ is divisible by $t^{\text{srank}(\lambda)}$.*

The following conjecture has been computationally verified (using Stembridge's SF Package for Maple [10]) for all partitions $\lambda \vdash n$ for $1 \leq n \leq 12$.

Conjecture 6.4. *The terms of lowest degree in Q_{λ} have degree exactly $\text{srank}(\lambda)$.*

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